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The stability of the Kronecker product of Schur functions

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ABSTRACT

In the late 1930s Murnaghan discovered the existence of a stabilization phenomenon for the Kronecker product of Schur functions. For n sufficiently large, the values of the Kronecker coefficients appearing in the product of two Schur functions of degree n do not depend on the first part of the indexing partitions, but only on the values of their remaining parts. We compute the exact value of n for which all the coefficients of a Kronecker product of Schur functions stabilize. We also compute two new bounds for the stabilization of a sequence of coefficients and show that they improve existing bounds of M. Brion and E. Vallejo.

Introduction

The understanding of the *Kronecker coefficients of the symmetric group* (the multiplicities appearing when the tensor product of two irreducible representations of the symmetric group is decomposed into irreducibles; equivalently, the structural constants for the Kronecker product of symmetric functions in the Schur basis) is a longstanding open problem. Richard Stanley writes "One of the main problems in the combinatorial representation theory of the symmetric group is to obtain a combinatorial interpretation for the Kronecker coefficients" [21]. It is also a source of new challenges such as the problem of describing the set of nonzero Kronecker coefficients [19], a problem inherited from quantum information theory [12,6]. Or proving that the positivity of a Kronecker coefficient can be

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decided in polynomial time, a problem posed by Mulmuley at the heart of his Geometric Complexity Theory [16].

The present work is part of a series of articles that study another family of nonnegative constants, the *reduced Kronecker coefficients* $\overline{g}_{\mu,\nu}^{\lambda}$, as a way to gain understanding about the Kronecker coefficients $g_{\mu,\nu}^{\lambda}$ [4,3]. In [3], we obtained the first explicit piecewise quasipolynomial description of a non-trivial family of Kronecker coefficients, the Kronecker coefficients indexed by two two-row shapes. This new description allowed us to test several conjectures of Mulmuley. As a result, we found a counterexample [4] for the strong version of his SH conjecture [16] on the behavior of the Kronecker coefficients under stretching of its indices.

The starting point of the investigation presented in this paper is a remarkable stability property for the Kronecker products of Schur functions discovered by Murnaghan [17,18]. This property is best shown on an example, that will be followed by a precise statement. Denote the Kronecker product of s_λ and s_β by $s_\lambda * s_\beta$. Then

$$\begin{aligned}
s_{2,2} * s_{2,2} &= s_4 + s_{1,1,1,1} + s_{2,2},\\
s_{3,2} * s_{3,2} &= s_5 + s_{2,1,1,1} + s_{3,2} + s_{4,1} + s_{3,1,1} + s_{2,2,1},\\
s_{4,2} * s_{4,2} &= s_6 + s_{3,1,1,1} + 2s_{4,2} + s_{5,1} + s_{4,1,1} + 2s_{3,2,1} + s_{2,2,2},\\
s_{5,2} * s_{5,2} &= s_7 + s_{4,1,1,1} + 2s_{5,2} + s_{6,1} + s_{5,1,1} + 2s_{4,2,1} + s_{3,2,2} + s_{4,3} + s_{3,3,1},\\
s_{6,2} * s_{6,2} &= s_8 + s_{5,1,1,1} + 2s_{6,2} + s_{7,1} + s_{6,1,1} + 2s_{5,2,1} + s_{4,2,2} + s_{5,3} + s_{4,3,1} + s_{4,4},\\
s_{7,2} * s_{7,2} &= s_9 + s_{6,1,1,1} + 2s_{7,2} + s_{8,1} + s_{7,1,1} + 2s_{6,2,1} + s_{5,2,2} + s_{6,3} + s_{5,3,1} + s_{5,4}.
\end{aligned}$$

Indeed, for all partitions of weight greater or equal to 8, we have

$$s_{\bullet,2} * s_{\bullet,2} = s_\bullet + s_{\bullet,1,1,1} + 2s_{\bullet,2} + s_{\bullet,1} + s_{\bullet,1,1} + 2s_{\bullet,2,1} + s_{\bullet,2,2} + s_{\bullet,3} + s_{\bullet,3,1} + s_{\bullet,4}.$$

Let $\alpha = (\alpha_1, \ldots, \alpha_k)$ be a partition and n an integer, define $\alpha[n] = (n - |\alpha|, \alpha_1, \ldots, \alpha_k)$. Murnaghan theorem says that for n big enough the expansions of $s_{\alpha[n]} * s_{\beta[n]}$ in the Schur basis all coincide, except for the first part of the indexing partitions (determined by the degree n). Therefore, given any three partitions α, β and γ, the sequence with general term $g_{\alpha[n]\beta[n]}^{\gamma[n]}$ is eventually constant. The reduced Kronecker coefficient $\overline{g}_{\alpha,\beta}^{\gamma}$ is defined as the stable value of this sequence. In our example, we see that $\overline{g}_{(2),(2)}^{(2)} = 2$ and $\overline{g}_{(2),(2)}^{(4)} = 1$.

In view of the difficulty of studying the Kronecker coefficients, it is surprising to obtain theorems that hold in general. Regardless of this, we present new results of general nature. We find an elegant formula that tells the point $n = \mathrm{stab}(\alpha, \beta)$ at which the expansion of the Kronecker product $s_{\alpha[n]} * s_{\beta[n]}$ stabilizes:

$$\mathrm{stab}(\alpha, \beta) = |\alpha| + |\beta| + \alpha_1 + \beta_1.$$

We also find new upper bounds for the point at which the sequence $g_{\alpha[n]\beta[n]}^{\gamma[n]}$ becomes constant, improving previously known bounds due to Brion [5] and Vallejo [25]. Interestingly, our investigations reduce to maximizing or bounding linear forms on the sets $\mathrm{Supp}(\alpha, \beta)$ of partitions γ such that $\overline{g}_{\alpha,\beta}^{\gamma} > 0$, where α and β are fixed partitions. This connects our research to a current problem of major importance: to describe the cones generated by the indices of the nonzero Kronecker coefficients [12,19]. Moreover, using Weyl's inequalities for eigenvalues of triples of Hermitian matrices [26], we find the maximum of γ_1 and upper bounds for all parts γ_k, among all γ in $\mathrm{Supp}(\alpha, \beta)$.

This paper is organized as follows, in Section 1 we give a detailed description of the main results of this work. In Section 2, we prove the theorem that allows us to recover the Kronecker coefficients from the reduced Kronecker coefficients. We also give an expression of the reduced Kronecker coefficients in terms of Littlewood–Richardson coefficients and Kronecker coefficients. The main significance of this expression is that it doesn't involve cancellations and it provides us with a tool to prove most of our main results. In Section 3, we provide a proof for the sharp bound for the stability of the Kronecker product. In the next section, Section 4, we consider the problem of finding bounds on the rows of γ, whenever $\bar{g}^{\gamma}_{\alpha,\beta} > 0$. We prove a theorem for a general upper bound for all rows of γ. From this theorem, we deduce a sharp bound for γ_1. In Section 5, we describe a general technique for deriving upper bounds for the stabilization of sequences of coefficients. Using this technique we get two new bounds. We show that one of these bounds improves the bounds of Brion and Vallejo. Finally, we compare our results to existing results in the literature.

1. Preliminaries and main results

Let λ be a partition (weakly decreasing sequences of positive integers) of n. Denote by V_λ the irreducible representation of the symmetric group $\mathfrak{S}_n$ indexed by λ. The Kronecker coefficient $g^{\lambda}_{\mu,\nu}$ is the multiplicity of V_λ in the decomposition into irreducible representations of the tensor product $V_\mu \otimes V_\nu$. The Frobenius map identifies the irreducible representations V_λ of the symmetric group with the Schur function s_λ. In doing so, it allows us to lift the tensor product of representations of the symmetric group to the setting of symmetric functions. Accordingly, the Kronecker coefficients $g^{\lambda}_{\mu,\nu}$ define the Kronecker product on symmetric functions by setting

$$s_\mu * s_\nu = \sum_\lambda g^{\lambda}_{\mu,\nu} s_\lambda.$$

The reader is referred to [14, Chapter I] or [21, Chapter 7] for the standard facts in the theory of symmetric functions.

Throughout this paper we follow the standard notation for partitions found in [14]. If $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ is a partition, its *parts* are its terms λ_i. The *weight* of λ is defined to be the sum of its parts, and it is denoted by $|\lambda|$. The number k of (nonzero) parts of λ is called its *length*, and denoted by $\ell(\lambda)$.

We identify a partition λ with its Ferrers diagram

$$D(\lambda) = \bigl\{(i, j)\colon\ 1 \leqslant i \leqslant \lambda_j,\ 1 \leqslant j \leqslant \ell(\lambda)\bigr\} \subseteq \mathbb{N}^2.$$

This way, we obtain that $\alpha \cap \beta = (\min(\alpha_1, \beta_1), \min(\alpha_2, \beta_2), \dots)$. The sum of two partitions $\alpha + \beta$ is defined as $(\alpha_1 + \beta_1, \alpha_2 + \beta_2, \dots)$.

Listing the number of points in each column of $D(\lambda)$ gives the transpose partition of λ, denoted by λ'; equivalently, one obtains the Ferrers diagram of λ' by reflecting the one of λ along its main diagonal.

The skew shape μ/ν is defined as the set difference $D(\mu) \setminus D(\nu)$. Notice that $D(\mu) \subset D(\lambda)$ if $\mu_i \leqslant \lambda_i$ for all i. Again, the intersection and union of skew-shapes is defined as the corresponding operations on their diagrams. The *width* of μ/ν is defined as the number of nonzero columns of μ/ν in $\mathbb{N}^2$.

Consider a partition λ and an integer n. Then $\bar{\lambda}$ is defined to be the partition $(\lambda_2, \lambda_3, \dots)$ and $\lambda[n]$ as the sequence

$$\bigl(n - |\lambda|, \lambda_1, \lambda_2, \dots\bigr).$$

Notice that $\lambda[n]$ is a partition only if $n - |\lambda| \geqslant \lambda_1$.

We extend the definition of a Schur function s_μ to the case where μ is any finite sequence of n integers. For this, we use the Jacobi–Trudi determinant,

$$s_\mu = \det(h_{\mu_i+j-i})_{1\leqslant i,j\leqslant n}, \tag{1}$$

where h_k is the complete homogeneous symmetric function of degree k. In particular, $h_k = 0$ if k is negative, and $h_0 = 1$. It is not hard to see that such a Jacobi–Trudi determinant s_μ is either zero or ± 1 times a Schur function (indexed by a partition).

Murnaghan theorem. *(See Murnaghan [17,18].) There exists a family of nonnegative integers ($\bar{g}_{\alpha\beta}^{\gamma}$) indexed by triples of partitions (α,β,γ) such that, for α and β fixed, only finitely many terms $\bar{g}_{\alpha\beta}^{\gamma}$ are nonzero, and for all $n \geqslant 0$,*

$$s_{\alpha[n]} * s_{\beta[n]} = \sum_\gamma \bar{g}_{\alpha\beta}^{\gamma} s_{\gamma[n]}. \tag{2}$$

Moreover, the coefficient $\bar{g}_{\alpha\beta}^{\gamma}$ vanishes unless the weights of the three partitions fulfill the inequalities:

$$|\alpha| \leqslant |\beta| + |\gamma|, \qquad |\beta| \leqslant |\alpha| + |\gamma|, \qquad |\gamma| \leqslant |\alpha| + |\beta|.$$

In what follows, we refer to these inequalities as *Murnaghan's inequalities* and we will denote $\mathrm{Supp}(\alpha,\beta)$ the set of all partitions γ such that $\bar{g}_{\alpha,\beta}^{\gamma} > 0$. We follow Klyachko [12] and call the coefficients $\bar{g}_{\alpha\beta}^{\gamma}$ the *reduced Kronecker coefficients*. An elegant proof of Murnaghan theorem, using vertex operators on symmetric functions, is given in [24].

Example 1. According to Murnaghan theorem the reduced Kronecker coefficients determine the Kronecker product of two Schur functions, even for small values of n. For instance,

$$s_{2,2} * s_{2,2} = s_4 + s_{1,1,1,1} + 2s_{2,2} + s_{3,1} + s_{2,1,1} + 2s_{1,2,1} + s_{0,2,2} + s_{1,3} + s_{0,3,1} + s_{0,4}.$$

The Jacobi–Trudi determinants corresponding to $s_{1,2,1}$ and $s_{0,2,2}$ have a repeated column, hence they are zero. On the other hand, it is easy to see that $s_{1,3} = -s_{2,2}$, $s_{0,3,1} = -s_{2,1,1}$, and $s_{0,4} = -s_{3,1}$. After taking into account the resulting cancellations, we recover the expression of the Kronecker product $s_{2,2} * s_{2,2}$ in the Schur basis: $s_4 + s_{1,1,1,1} + s_{2,2}$.

The reduced Kronecker coefficients contain the Littlewood–Richardson coefficients as special cases.

Murnaghan–Littlewood theorem. *(See Murnaghan [18], Littlewood [13].) Let α, β and γ be partitions. If $|\gamma| = |\alpha| + |\beta|$, then the reduced Kronecker coefficient $\bar{g}_{\alpha,\beta}^{\gamma}$ is equal to the Littlewood–Richardson coefficient $c_{\alpha,\beta}^{\gamma}$.*

Finally, a remarkable result of Christandl, Harrow, and Mitchison (originally stated for the Kronecker coefficients) says that the set

$$\mathrm{RKron}_k = \left\{(\alpha,\beta,\gamma) \;\middle|\; \ell(\alpha),\ell(\beta),\ell(\gamma) \leqslant k \text{ and } \bar{g}_{\alpha,\beta}^{\gamma} > 0\right\}$$

is a finitely generated semigroup under componentwise addition [6]. That is, if $\bar{g}_{\alpha,\beta}^{\gamma} \neq 0$ and $\bar{g}_{\hat\alpha\hat\beta}^{\hat\gamma} \neq 0$, then $\bar{g}_{\alpha+\hat\alpha,\beta+\hat\beta}^{\gamma+\hat\gamma} \neq 0$. This implies that RKron_k is closed under stretching. That is, that $\bar{g}_{\alpha,\beta}^{\gamma} \neq 0$ implies that $\bar{g}_{N\alpha,N\beta}^{N\gamma} \neq 0$ for all $N > 0$.

Both Klyachko and Kirillov have conjectured that the converse also holds. That is to say, that the reduced Kronecker coefficients satisfy the saturation property [12,10]. Remarkably, the Kronecker coefficients do not satisfy the saturation property. For example,

$$g_{(n,n),(n,n)}^{(n,n)} = 0 \quad \text{if } n \text{ is odd}, \quad \text{but} \quad g_{(n,n),(n,n)}^{(n,n)} = 1 \quad \text{if } n \text{ is even}.$$

At this point, we hope that the reader is convinced that the reduced Kronecker coefficients are interesting objects on their own.

We are ready to describe the results of this article. In Theorem 1.1 we give an explicit formula for recovering the value of the Kronecker coefficients from the reduced Kronecker coefficients. Let $u = (u_1, u_2, \ldots)$ be an infinite sequence and i a positive integer. Define $u^{\dagger i}$ as the sequence obtained from u by adding 1 to its $i-1$ first terms and erasing its i-th term:

$$u^{\dagger i} = (1 + u_1, 1 + u_2, \ldots, 1 + u_{i-1}, u_{i+1}, u_{i+2}, \ldots).$$

Partitions are identified with infinite sequences by appending trailing zeros. Under this identification, when λ is a partition then so is $\lambda^{\dagger i}$ for all positive i, but in general they are not partitions of the same number.

Theorem 1.1 *(Computing the Kronecker coefficients from the reduced Kronecker coefficients). Let n be a non-negative integer and λ, μ, and ν be partitions of n. Then*

$$g_{\mu\nu}^{\lambda} = \sum_{i=1}^{\ell(\mu)\ell(\nu)} (-1)^{i+1} \bar{g}_{\bar{\mu}\bar{\nu}}^{\lambda^{\dagger i}}. \tag{3}$$

This theorem was stated without proof in [3], and used to compute an explicit piecewise quasipolynomial description for the Kronecker coefficients indexed by two two-row shapes.

Murnaghan theorem implies the stability property for the Kronecker products $s_{\alpha[n]} * s_{\beta[n]}$ presented in the introduction. Indeed, for n big enough, all sequences $\gamma[n]$ for $\gamma \in \mathrm{Supp}(\alpha, \beta)$ are partitions, and then (2) is the expansion of $s_{\alpha[n]} * s_{\beta[n]}$ in the Schur basis. In particular, for n big enough, the Kronecker coefficient $g_{\alpha[n],\beta[n]}^{\gamma[n]}$ is equal to the reduced Kronecker coefficient $\bar{g}_{\alpha,\beta}^{\gamma}$. It is natural to ask about the index n at which the expansion of $s_{\alpha[n]} * s_{\beta[n]}$ stabilizes. This index is defined as follows.

Definition ($\mathrm{stab}(\alpha, \beta)$). Let V be the linear operator on symmetric functions defined on the Schur basis by: $V(s_\lambda) = s_{\lambda+(1)}$ for all partitions λ. Let α and β be partitions. Then $\mathrm{stab}(\alpha, \beta)$ is defined as the smallest integer n such that $s_{\alpha[n+k]} * s_{\beta[n+k]} = V^k(s_{\alpha[n]} * s_{\beta[n]})$ for all $k > 0$.

As an illustration see the example in the introduction, there $\alpha = \beta = (2)$ and the Kronecker product is stable starting at $s_{(6,2)} * s_{(6,2)}$. Since $(6, 2)$ is a partition of 8, we get that $\mathrm{stab}(\alpha, \beta) = 8$.

Theorem 1.2. *Let α and β be two partitions. Then*

$$\mathrm{stab}(\alpha, \beta) = |\alpha| + |\beta| + \alpha_1 + \beta_1.$$

In order to show that this theorem holds, we first reduce the calculation of $\mathrm{stab}(\alpha, \beta)$ to maximizing a linear form on $\mathrm{Supp}(\alpha, \beta)$ (Lemma 3.1):

$$\mathrm{stab}(\alpha, \beta) = \max\{|\gamma| + \gamma_1 \mid \gamma \text{ partition}, \bar{g}_{\alpha,\beta}^{\gamma} > 0\}.$$

Then, we show that (Theorem 3.2)

$$\max\{|\gamma|+\gamma_1 \mid \gamma \text{ partition}, \bar{g}^{\gamma}_{\alpha,\beta}>0\}=|\alpha|+|\beta|+\alpha_1+\beta_1 \tag{4}$$

using a decomposition of $\bar{g}^{\gamma}_{\alpha\beta}$ as a sum of nonnegative summands derived from Murnaghan's theorem. This decomposition is described in Lemma 2.1.

We also obtain other interesting bounds for linear forms over the set $\mathrm{Supp}(\alpha,\beta)$. In Theorem 4.1 we show that

$$\max\{\gamma_1 \mid \gamma \text{ partition}, \bar{g}^{\gamma}_{\alpha,\beta}>0\}=|\alpha\cap\beta|+\max(\alpha_1,\beta_1). \tag{5}$$

More generally we obtain in Theorem 4.3 that, whenever $\bar{g}^{\gamma}_{\alpha,\beta}>0$, we have for all positive integers i, j:

$$\gamma_{i+j-1}\leqslant |E_i\alpha\cap E_j\beta|+\alpha_i+\beta_j$$

where $E_k\lambda$ stands for the partition obtained from λ by erasing its k-th part.

We also obtain (Theorem 4.4):

$$\max\{|\gamma| \mid \gamma \text{ partition}, \bar{g}^{\gamma}_{\alpha,\beta}>0\}=|\alpha|+|\beta|,$$
$$\min\{|\gamma| \mid \gamma \text{ partition}, \bar{g}^{\gamma}_{\alpha,\beta}>0\}=\max(|\alpha|,|\beta|)-|\alpha\cap\beta|.$$

Note that formula (5) is reminiscent to the following result for the Kronecker coefficients:

Proposition 1.3. *(See Klemm [11], Dvir [8, Theorem 1.6], Clausen and Meier [7, Satz 1.1].) Let α and β be partitions with the same weight. Then*

$$\max\{\gamma_1 \mid \gamma \textit{ partition s.t. } g^{\gamma}_{\alpha,\beta}>0\}=|\alpha\cap\beta|.$$

To conclude this work, in Section 5 we consider a weaker version of the stabilization problem (think uniform convergence vs. simple convergence). As mentioned, Murnaghan theorem also implies that each particular sequence of Kronecker coefficients $g^{\gamma[n]}_{\alpha[n],\beta[n]}$ stabilizes with value $\bar{g}^{\gamma}_{\alpha,\beta}$, possibly before reaching $\mathrm{stab}(\alpha,\beta)$. More is known about these sequences:

Monotonicity theorem. *(See Brion [5], see also [15].) Let α, β and γ be partitions. The sequence with general term $g^{\gamma[n]}_{\alpha[n],\beta[n]}$ is weakly increasing.*

Definition *($\mathrm{stab}(\alpha,\beta,\gamma)$).* Let α, β, γ be partitions. Then $\mathrm{stab}(\alpha,\beta,\gamma)$ is defined as the smallest integer N such that the sequences $\alpha[N]$, $\beta[N]$ and $\gamma[N]$ are partitions and $g^{\gamma[n]}_{\alpha[n],\beta[n]}=\bar{g}^{\gamma}_{\alpha,\beta}$ for all $n\geqslant N$.

Lemma 5.1 describes a general technique for producing linear upper bounds for $\mathrm{stab}(\alpha,\beta,\gamma)$ from any linear function f such that $\gamma_1\leqslant f(\alpha,\beta,\bar{\gamma})$ whenever $\bar{g}^{\gamma}_{\alpha,\beta}>0$. This method provides two new upper bounds N_1 and N_2 for $\mathrm{stab}(\alpha,\beta,\gamma)$.

The first bound is found by applying Lemma 5.1 to the bound (5) for γ_1 obtained in Theorem 4.1.

Theorem 1.4. *Let $M_1(\alpha,\beta;\gamma)=|\gamma|+|\bar{\alpha}\cap\bar{\beta}|+\alpha_1+\beta_1$ and*

$$N_1(\alpha,\beta,\gamma)=\min\{M_1(\alpha,\beta;\gamma),M_1(\alpha,\gamma;\beta),M_1(\beta,\gamma;\alpha)\}.$$

Then $\mathrm{stab}(\alpha,\beta,\gamma)\leqslant N_1(\alpha,\beta,\gamma)$.

The second bound is obtained by applying Lemma 5.1 to the bound (4) obtained in Theorem 3.2.

Theorem 1.5. *Let*

$$N_2(\alpha, \beta, \gamma) = \left[\frac{|\alpha| + |\beta| + |\gamma| + \alpha_1 + \beta_1 + \gamma_1}{2}\right]$$

where $[x]$ *denotes the integer part of x. Then* $\operatorname{stab}(\alpha, \beta, \gamma) \leqslant N_2(\alpha, \beta, \gamma)$.

We finish our work by placing the new bounds in the context of the current literature. We show in Proposition 5.2 that N_1 beats those of Ernesto Vallejo [25] and Michel Brion [5]. But neither N_1 nor N_2 is better than the other. There are infinite families of examples where $N_1 < N_2$ (see Example 5 on the Kronecker coefficients indexed by three hooks), and others where $N_2 < N_1$ (see Example 6 on the Kronecker coefficients indexed by two two-row shapes). Finally, we revisit the work of Rosas [20], Ballantine and Orellana [2], and [3] where the situation for some restricted families of Kronecker coefficients is addressed.

2. The reduced Kronecker coefficients

In this section we show how to recover the Kronecker coefficients from the knowledge of the reduced Kronecker coefficients. We also present an expression for the reduced Kronecker coefficients as sums of nonnegative terms, involving Littlewood–Richardson coefficients as well as Kronecker coefficients, that will be useful in the next two sections.

We denote by $\langle \mid \rangle$ the Hall inner product on symmetric functions. Recall that formula (3) in Theorem 1.1 shows that we can recover the Kronecker coefficients from the reduced ones:

$$g_{\mu\nu}^{\lambda} = \sum_{i=1}^{\ell(\mu)\ell(\nu)} (-1)^{i+1} \bar{g}_{\bar{\mu}\bar{\nu}}^{\lambda^{\dagger i}}.$$

We now provide the proof.

Proof of Theorem 1.1. Murnaghan theorem tells us that

$$s_\mu * s_\nu = \sum_{\gamma \in \operatorname{Supp}(\bar{\mu}, \bar{\nu})} \bar{g}_{\bar{\mu}\bar{\nu}}^{\gamma} s_{\gamma[n]}.$$

Performing the scalar product with s_λ in the preceding equation yields

$$g_{\mu,\nu}^{\lambda} = \sum_{\gamma \in \operatorname{Supp}(\bar{\mu}, \bar{\nu})} \bar{g}_{\bar{\mu}\bar{\nu}}^{\gamma} \langle s_{\gamma[n]} \mid s_\lambda \rangle. \tag{6}$$

Let $\gamma \in \operatorname{Supp}(\bar{\mu}, \bar{\nu})$ be a partition such that $\langle s_{\gamma[n]} \mid s_\lambda \rangle \neq 0$, and let $k = \ell(\gamma)$. Then λ has length at most $k+1$ and the Jacobi–Trudi determinants $s_{\gamma[n]}$ and s_λ have the same columns, up to the order, see Eq. (1). That is, the sequence

$$v = \bigl(n - |\gamma|, \gamma_1, \gamma_2, \ldots, \gamma_k\bigr) + (k+1, k, k-1, \ldots, 1)$$

is a permutation of the decreasing sequence $u = \lambda + (k+1, k, k-1, \ldots, 1)$. (As usual one sets $\lambda_j = 0$ for $j > \ell(\lambda)$.)

By construction, we have that v is decreasing starting at v_2. Therefore, there exists an index i such that $u_j = v_j + 1$ for all $j < i$ and $u_j = v_j$ for all $j > i$. This means that $\gamma = \lambda^{\dagger i}$ for some $i \leqslant k+1$. Since $\gamma \in \mathrm{Supp}(\bar\mu, \bar\nu)$ we conclude that $k = \ell(\gamma) \leqslant \ell(\mu)\ell(\nu) - 1$ and thus $i \leqslant \ell(\gamma)$.

Finally $\langle s_{\gamma[n]} \mid s_\lambda \rangle$ is the sign of the permutation that transforms v into the decreasing sequence u. This permutation is the cycle $(i, i-1, \ldots, 2, 1)$, which has sign $(-1)^{i+1}$. This shows that only the partitions $\gamma = \lambda^{\dagger i}$, for i between 1 and $\ell(\mu)\ell(\nu)$, contribute to the sum in the right-hand side of (6), and that the contribution of $\lambda^{\dagger i}$ is $(-1)^{i+1}\bar g^{\gamma}_{\bar\mu\bar\nu}$. $\square$

The operator on symmetric functions $f \mapsto f^\perp$ is defined as the operator dual to multiplication with respect to the inner product, $\langle\,|\,\rangle$.

Define $c^{\delta}_{\alpha,\beta,\gamma}$ as the coefficients of s_δ in $s_\alpha s_\beta s_\gamma$. From the definition of the Littlewood–Richardson coefficients as the structural constant for the product of two Schur functions, we immediately obtain that

$$c^{\delta}_{\alpha,\beta,\gamma} = \sum_{\varphi} c^{\varphi}_{\alpha,\beta} c^{\delta}_{\varphi,\gamma}. \tag{7}$$

Lemma 2.1. *Let α, β, γ be partitions. Then $\bar g^{\gamma}_{\alpha,\beta}$ is positive if and only if there exist partitions δ, ϵ, ζ, ρ, σ, τ such that all four coefficients $g^{\zeta}_{\delta,\epsilon}$, $c^{\alpha}_{\delta,\sigma,\tau}$, $c^{\beta}_{\epsilon,\rho,\tau}$ and $c^{\gamma}_{\zeta,\rho,\sigma}$ are positive. Moreover,*

$$\bar g^{\gamma}_{\alpha,\beta} = \sum g^{\zeta}_{\delta,\epsilon} c^{\alpha}_{\delta,\sigma,\tau} c^{\beta}_{\epsilon,\rho,\tau} c^{\gamma}_{\zeta,\rho,\sigma}. \tag{8}$$

Proof. Given partitions α and β, define the following symmetric function

$$R_{\alpha,\beta} = \sum_{\delta,\epsilon,\tau} \big((s_\delta s_\tau)^\perp s_\alpha\big)\big((s_\epsilon s_\tau)^\perp s_\beta\big)(s_\delta * s_\epsilon)$$

where the sum is over all triples of partitions δ, ϵ, τ. For n integer, let U_n be the linear operator on symmetric functions that sends the Schur function s_λ to the Jacobi–Trudi determinant $s_{\lambda[n]}$. Littlewood showed in [13] that for all partitions α and β and all integers n,

$$s_{\alpha[n]} * s_{\beta[n]} = U_n R_{\alpha,\beta}. \tag{9}$$

Formula (9) is also presented in [1, formula 6.1] and [23, formula 8].

Comparing (9) with Murnaghan theorem we see that

$$U_n R_{\alpha,\beta} = U_n \sum_{\gamma} \bar g^{\gamma}_{\alpha,\beta} s_\gamma.$$

The operator U_n is not injective, but its restriction to the symmetric functions of degree at most $n/2$ is. Indeed, when $|\gamma| \leqslant n/2$, the sequence $\gamma[n]$ is a partition. Therefore, taking n big enough we can deduce that $R_{\alpha,\beta} = \sum_\gamma \bar g^{\gamma}_{\alpha,\beta} s_\gamma$.

Let us determine the expansion $\sum_\gamma r^{\gamma}_{\alpha,\beta} s_\gamma$ of $R_{\alpha,\beta}$ in the Schur basis. We have

$$(s_\delta s_\tau)^\perp s_\alpha = \sum_{\sigma} c^{\alpha}_{\delta,\sigma,\tau} s_\sigma,$$

$$(s_\epsilon s_\tau)^\perp s_\beta = \sum_{\rho} c^{\beta}_{\epsilon,\rho,\tau} s_\rho,$$

$$s_\delta * s_\epsilon = \sum_{\zeta} g^{\zeta}_{\delta,\epsilon} s_\zeta.$$

Therefore,

$$\begin{aligned} R_{\alpha,\beta} &= \sum g^{\zeta}_{\delta,\epsilon} c^{\alpha}_{\delta,\sigma,\tau} c^{\beta}_{\epsilon,\rho,\tau} s_\sigma s_\rho s_\tau \\ &= \sum g^{\zeta}_{\delta,\epsilon} c^{\alpha}_{\delta,\sigma,\tau} c^{\beta}_{\epsilon,\rho,\tau} c^{\gamma}_{\sigma,\rho,\tau} s_\gamma. \end{aligned}$$

We obtain Eq. (8). □

3. Stability: The Kronecker product

In this section we consider the stability of the Kronecker product of Schur functions. We provide a proof for Theorem 1.2 which provides a sharp bound for this stability.

Lemma 3.1. *Let α and β be partitions. Then*

$$\mathrm{stab}(\alpha,\beta) = \max\{|\gamma| + \gamma_1 \mid \gamma \textit{ partition}, \bar{g}^{\gamma}_{\alpha,\beta} > 0\}.$$

Proof. Let $N = \max\{|\gamma| + \gamma_1 \mid \gamma$ partition, $\bar{g}^{\gamma}_{\alpha,\beta} > 0\}$. If α and β are equal to the empty partition then $N = 0 = \mathrm{stab}(\alpha,\beta)$. In the other cases, that we consider now, we have $N > 0$.

Remember (from the definition of $\mathrm{stab}(\alpha,\beta)$ in Section 1) that V is the linear operator that fulfills $V(s_\lambda) = s_{\lambda+(1)}$ for all partitions λ. For all $\gamma \in \mathrm{Supp}(\alpha,\beta)$ and $k > 0$, the sequences $\gamma[N]$ and $\gamma[N+k]$ are partitions, therefore $s_{\gamma[N+k]} = V^k(s_{\gamma[N]})$. Using Murnaghan theorem, we deduce that

$$s_{\alpha[N]} * s_{\beta[N]} = \sum_{\gamma \in \mathrm{Supp}(\alpha,\beta)} \bar{g}^{\gamma}_{\alpha\beta} s_{\gamma[N]},$$

$$s_{\alpha[N+k]} * s_{\beta[N+k]} = \sum_{\gamma \in \mathrm{Supp}(\alpha,\beta)} \bar{g}^{\gamma}_{\alpha\beta} s_{\gamma[N+k]}.$$

We obtain that

$$s_{\alpha[N+k]} * s_{\beta[N+k]} = V^k(s_{\alpha[N]} * s_{\beta[N]}).$$

This proves that $N \geqslant \mathrm{stab}(\alpha,\beta)$.

The equality will be obtained by proving additionally that $N - 1 < \mathrm{stab}(\alpha,\beta)$. Since N is defined as the max for $|\gamma| + \gamma_1$ for those partitions γ where $g^{\gamma}_{\alpha,\beta} > 0$, there exists a partition $\gamma \in \mathrm{Supp}(\alpha,\beta)$ such that $|\gamma| + \gamma_1 = N$. Therefore, the first and the second parts of $\gamma[N]$ are equal. This shows that $s_{\gamma[N]}$ is not in the image of V. It follows that $s_{\alpha[N]} * s_{\beta[N]}$ is not in the image of V. In particular, $s_{\alpha[N]} * s_{\beta[N]}$ is not equal to $V(s_{\alpha[N-1]} * s_{\beta[N-1]})$. □

Theorem 3.2. *Let α, β be partitions. Then,*

$$\max\{|\gamma| + \gamma_1 \mid \gamma \textit{ partition}, \bar{g}^{\gamma}_{\alpha,\beta} > 0\} = |\alpha| + |\beta| + \alpha_1 + \beta_1. \tag{4}$$

Proof. Let γ be a partition such that $\bar{g}^{\gamma}_{\alpha,\beta} > 0$. By Lemma 2.1, there exist partitions δ, ϵ, ζ, ρ, σ, τ such that all four coefficients $g^{\zeta}_{\delta,\epsilon}$, $c^{\alpha}_{\delta,\sigma,\tau}$, $c^{\beta}_{\epsilon,\rho,\tau}$ and $c^{\gamma}_{\zeta,\rho,\sigma}$ are positive.

The Littlewood–Richardson rule together with Eq. (7) implies that if $c^{\gamma}_{\zeta,\rho,\sigma} > 0$ then $\gamma_1 \leqslant \zeta_1 + \rho_1 + \sigma_1$. Since $c^{\gamma}_{\zeta,\rho,\sigma} > 0$, we have also $|\gamma| = |\zeta| + |\rho| + |\sigma|$. Therefore $|\gamma| + \gamma_1 \leqslant |\zeta| + \zeta_1 + |\rho| + \rho_1 + |\sigma| + \sigma_1$. Obviously $\zeta_1 \leqslant |\zeta|$. Thus

$$|\gamma| + \gamma_1 \leqslant 2|\zeta| + |\rho| + \rho_1 + |\sigma| + \sigma_1. \tag{10}$$

Since $g^{\zeta}_{\delta,\epsilon} > 0$ we have $|\zeta| = |\delta| = |\epsilon|$. Replacing $2|\zeta|$ with $|\delta| + |\epsilon|$ in (10) yields

$$|\gamma| + \gamma_1 \leqslant |\delta| + |\sigma| + \sigma_1 + |\epsilon| + |\rho| + \rho_1. \tag{11}$$

Since $c^{\alpha}_{\delta,\sigma,\tau} > 0$ we have $\sigma \subset \alpha$ and thus $\sigma_1 \leqslant \alpha_1$. We have also $|\delta| + |\sigma| \leqslant |\alpha|$. Therefore $|\delta| + |\sigma| + \sigma_1 \leqslant |\alpha| + \alpha_1$.

Similarly, $c^{\beta}_{\epsilon,\rho,\tau} > 0$ implies $|\epsilon| + |\rho| + \rho_1 \leqslant |\beta| + \beta_1$.

Substituting these two new inequalities in (11) provides the following inequality

$$|\gamma| + \gamma_1 \leqslant |\alpha| + |\beta| + \alpha_1 + \beta_1.$$

We now show that the bound is achieved. Consider the reduced Kronecker coefficient $\bar{g}^{\alpha+\beta}_{\alpha,\beta}$. The Murnaghan–Littlewood theorem implies that it is equal to the Littlewood–Richardson coefficient $c^{\alpha+\beta}_{\alpha,\beta}$ which is equal to 1. This proves that the upper bound $|\alpha| + |\beta| + \alpha_1 + \beta_1$ on $\mathrm{Supp}(\alpha,\beta)$, for $|\gamma| + \gamma_1$, is reached with $\gamma = \alpha + \beta$. $\square$

Theorem 1.2 is now a direct consequence of Lemma 3.1 and Theorem 3.2.

4. Bounds for linear forms on Supp($\boldsymbol{\alpha}, \boldsymbol{\beta}$)

In this section we provide proofs for the bounds of the lengths of the rows of γ when $\bar{g}^{\gamma}_{\alpha,\beta} > 0$. In particular, we provide a sharp bound for the first row and upper bounds for the remaining rows. Theorem 4.1 gives a first step towards describing the set partitions indexing the nonzero reduced Kronecker coefficients, that is $\mathrm{Supp}(\alpha,\beta)$.

Indeed, we show that

Theorem 4.1. *Let α and β be partitions, then*

$$\max\bigl\{\gamma_1 \bigm| \gamma \textit{ partition}, \bar{g}^{\gamma}_{\alpha,\beta} > 0\bigr\} = |\alpha \cap \beta| + \max(\alpha_1, \beta_1).$$

From Theorem 4.1, we obtain that given any three partitions μ, ν and λ of n such that $g^{\lambda}_{\mu,\nu} > 0$, we have that

$$\lambda_2 \leqslant \min\left(\frac{n}{2}, |\bar{\mu} \cap \bar{\nu}| + \max(\mu_2, \nu_2)\right).$$

Fix two partitions α and β. To prove Theorem 4.1 we first prove an upper bound for all the rows of γ whenever $\bar{g}^{\gamma}_{\alpha,\beta} > 0$ (Theorem 4.3).

For λ partition and k positive integer, set $E_k\lambda$ for the partition obtained from λ by erasing its k-th part (or leaving λ unchanged when it has less than k parts). In particular $E_1\lambda = \bar{\lambda}$.

Lemma 4.2. *Let α, δ, σ and τ be partitions such that $c^{\alpha}_{\delta,\sigma,\tau} > 0$. Let i be a positive integer. Then there exists a set A such that $D(\delta) \subset D(E_i\alpha) \cup A$ and $|A| + \sigma_k \leqslant \alpha_k$.*

Proof. By Eq. (7), there exists a partition κ such that $c^{\alpha}_{\kappa,\tau} > 0$ and $c^{\kappa}_{\delta,\sigma} > 0$ since $c^{\alpha}_{\delta,\sigma,\tau} > 0$. In particular $D(\delta) \subset D(\kappa) \subset D(\alpha)$.

Let $S_i = \{(x, y) \mid x \geqslant 1 \text{ and } y \geqslant i\}$ and let H be the set-theoretical difference $D(\delta) \setminus D(\bar{\kappa})$, where $\bar{\kappa}$ is the partition obtained from κ by deleting the first part.

Notice that H is a horizontal strip consisting of all boxes of $D(\delta)$ having no box of $D(\kappa)$ above them, see Fig. 1 for an example.

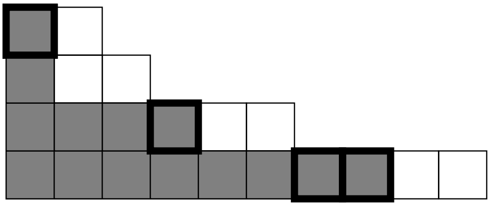

Fig. 1. The horizontal strip H (boxes with thick edges) for $\kappa = (10, 6, 3, 2)$ (white and grey boxes) and $\delta = (8, 4, 1, 1)$ (grey boxes).

Let $A = S_i \cap H$, notice that this is the horizontal strip contained in H strictly above the $(i-1)$-st row. We have

$$|A| = \kappa_i - \operatorname{width}\bigl(D(\kappa/\delta) \cap S_i\bigr).$$

On the other hand, since $c^{\kappa}_{\delta,\sigma} > 0$, there exists a Littlewood–Richardson tableau with shape κ/δ and content σ. In this tableau, there is at most one occurrence of i by column of κ/δ, and they are all in row i or higher. Therefore,

$$\sigma_i \leqslant \operatorname{width}\bigl(D(\kappa/\delta) \cap S_i\bigr).$$

As a consequence,

$$|A| + \sigma_i \leqslant \kappa_i.$$

Since $D(\kappa) \subset D(\alpha)$ we conclude that $|A| + \sigma_i \leqslant \alpha_i$.

Now by construction of A,

$$D(\delta) \cap S_i \subset \bigl(D(\overline{\kappa}) \cap S_i\bigr) \cup A$$

and clearly $D(\delta) \setminus S_i \subset D(\kappa) \setminus S_i$. Therefore

$$D(\delta) \subset \bigl(D(\kappa) \setminus S_i\bigr) \cup \bigl(D(\overline{\kappa}) \cap S_i\bigr) \cup A.$$

Finally, observe that $D(E_i\kappa) = (D(\kappa) \setminus S_i) \cup (D(\overline{\kappa}) \cap S_i)$. Therefore,

$$D(\delta) \subset D(E_i\kappa) \cup A.$$

Since $D(\kappa) \subset D(\alpha)$ we have $D(E_i\kappa) \subset D(E_i\alpha)$, and thus

$$D(\delta) \subset D(E_i\alpha) \cup A. \qquad \square$$

Theorem 4.3. *Let α, β and γ be partitions such that $\overline{g}^{\gamma}_{\alpha,\beta} > 0$ and let i, j and k be a positive integers such that $i + j - 1 = k$, then we have*

$$\gamma_k \leqslant |E_i\alpha \cap E_j\beta| + \alpha_i + \beta_j.$$

Proof. Let i and j be such that $i+j-1=k$.

By Lemma 2.1, there exist partitions δ, ϵ, ζ, ρ, σ, τ such that all four coefficients $g^{\zeta}_{\delta,\epsilon}$, $c^{\alpha}_{\delta,\sigma,\tau}$, $c^{\beta}_{\epsilon,\rho,\tau}$, $c^{\gamma}_{\zeta,\rho,\sigma}$ are positive.

By Eq. (7) and since $c^{\gamma}_{\zeta,\rho,\sigma}>0$, there exists a partition ϕ such that $c^{\gamma}_{\zeta,\phi}>0$ and $c^{\phi}_{\rho,\sigma}>0$. Weyl's inequalities for eigenvalues of Hermitian matrices ([26] or Eq. (2) in [9]) imply that whenever a Littlewood–Richardson coefficient $c^{\lambda}_{\mu,\nu}$ is nonzero there is $\lambda_{p+q-1}\leqslant\mu_p+\nu_q$ for all p, q (see Proposition 5 in [9]). Apply this to $c^{\gamma}_{\zeta,\phi}$ with $p=1$, $q=k$: we obtain $\gamma_k\leqslant\zeta_1+\phi_k$. Apply Weyl's inequalities to $c^{\phi}_{\rho,\sigma}$ with $p=j$, $q=i$: we obtain $\phi_k\leqslant\rho_j+\sigma_i$. It follows that $\gamma_k\leqslant\zeta_1+\sigma_i+\rho_j$.

Since $g^{\zeta}_{\delta,\epsilon}>0$, we have $\zeta_1\leqslant|\delta\cap\epsilon|$ by Proposition 1.3, then

$$\gamma_k\leqslant|\delta\cap\epsilon|+\rho_j+\sigma_i. \tag{12}$$

Since $c^{\alpha}_{\delta,\sigma,\tau}>0$, Lemma 4.2 implies that there exists a set A_1 such that

$$D(\delta)\subset D(E_i\alpha)\cup A_1\quad\text{and}\quad |A_1|+\sigma_i\leqslant\alpha_i.$$

Similarly for $c^{\beta}_{\epsilon,\rho,\tau}>0$, Lemma 4.2 implies that there exists a set A_2 such that

$$D(\epsilon)\subset D(E_j\beta)\cup A_2\quad\text{and}\quad |A_2|+\rho_j\leqslant\beta_j.$$

Therefore,

$$D(\delta\cap\epsilon)\subset D(E_i\alpha\cap E_j\beta)\cup A_1\cup A_2.$$

As a consequence,

$$|\delta\cap\epsilon|\leqslant|E_i\alpha\cap E_j\beta|+|A_1|+|A_2|.$$

This together with (12) yields

$$\gamma_k\leqslant|E_i\alpha\cap E_j\beta|+|A_1|+\sigma_i+|A_2|+\rho_j.$$

Remembering that $|A_1|+\sigma_i\leqslant\alpha_i$ and $|A_2|+\rho_j\leqslant\beta_j$, we get the claimed inequality. $\square$

Proof of Theorem 4.1. The bound holds by Theorem 4.3 since $|E_1\alpha\cap E_1\beta|+\alpha_1+\beta_1=|\alpha\cap\beta|+\max(\alpha_1,\beta_1)$. Let us now show it is reached. Choose $\delta=\epsilon=\overline{\alpha}\cap\overline{\beta}$ and for ζ a partition such that $g^{\zeta}_{\delta,\epsilon}>0$ and $\zeta_1=|\delta\cap\epsilon|=|\overline{\alpha}\cap\overline{\beta}|$, such a partition exists by Proposition 1.3.

Let τ be the empty partition, and choose σ as follows. First set $\sigma_1=\alpha_1$. This will ensure that $c^{\alpha}_{\delta,\sigma,\tau}=c^{\alpha}_{\delta,\sigma}=c^{\overline{\alpha}}_{\delta,\overline{\sigma}}$.

Since the Littlewood–Richardson coefficients $c^{\overline{\alpha}}_{\delta,\kappa}$ are the coefficients in the expansion of the nonzero skew-Schur function $s_{\overline{\alpha}/\delta}$ in the Schur basis, at least one of Schur function has to appear with nonzero coefficient. For $\overline{\sigma}$ choose one such partition κ (observe that $D(\kappa)\subset D(\overline{\alpha})$, therefore $\kappa_1\leqslant\alpha_2\leqslant\alpha_1=\sigma_1$). Define similarly ρ. Finally set $\gamma=\zeta+\sigma+\rho$. $\square$

Theorem 4.4 *(The maximum and minimum weight of partitions indexing nonzero reduced Kronecker coefficients). Let α and β be partitions. We have*

$$\max\{|\gamma| \mid \gamma \textit{ partition}, \bar{g}^{\gamma}_{\alpha,\beta} > 0\} = |\alpha| + |\beta|,$$
$$\min\{|\gamma| \mid \gamma \textit{ partition}, \bar{g}^{\gamma}_{\alpha,\beta} > 0\} = \max(|\alpha|, |\beta|) - |\alpha \cap \beta|.$$

Proof. We will show that the first equality is a consequence of Murnaghan's inequalities and the second of Proposition 1.3 (the inequalities of Klemm–Dvir–Clausen–Meier).

From Murnaghan's inequalities we know that $|\gamma| \leqslant |\alpha| + |\beta|$ for all $\gamma \in \mathrm{Supp}(\alpha, \beta)$. Moreover, this maximum is achieved, take $\gamma = \alpha + \beta$, then $c^{\alpha+\beta}_{\alpha,\beta} > 0$ and finally $\bar{g}^{\alpha+\beta}_{\alpha,\beta} = c^{\alpha+\beta}_{\alpha,\beta}$ by the theorem of Littlewood and Murnaghan.

To show the second bound, assume that $\bar{g}^{\gamma}_{\alpha,\beta} > 0$. There exists n such that $g^{\gamma[n]}_{\alpha[n],\beta[n]} = \bar{g}^{\gamma}_{\alpha,\beta}$. By Proposition 1.3 we have that $n - |\gamma| \leqslant |\alpha[n] \cap \beta[n]|$. Hence,

$$\bigl|\alpha[n] \cap \beta[n]\bigr| = \min(n - |\alpha|, n - |\beta|) + |\alpha \cap \beta| = n - \max(|\alpha|, |\beta|) + |\alpha \cap \beta|.$$

We conclude that $|\gamma| \geqslant \max(|\alpha|, |\beta|) - |\alpha \cap \beta|$.

Again by Proposition 1.3 we know that there is a partition γ for which $n - |\gamma| = |\alpha[n] \cap \beta[n]|$, hence this bound is sharp. $\square$

Corollary 4.5. *Let α and β be partitions and i and j positive integers such that $k = i + j - 1$. Then*

$$\max\{\gamma_k \mid \gamma \textit{ partition}, \bar{g}^{\gamma}_{\alpha,\beta} > 0\} \leqslant \min\left(|E_i\alpha \cap E_j\beta| + \alpha_i + \beta_j, \left[\frac{|\alpha| + |\beta|}{k}\right]\right).$$

Proof. This is a straightforward consequence of Theorems 4.3 and 4.4. $\square$

Example 2. Let $\alpha = (2)$ and $\beta = (4, 3, 2)$, then the first row of the table gives the nonzero values of γ_k and the second row gives the upper bounds given by Corollary 4.5.

k	1	2	3	4	5
max values for γ_k	6	4	3	2	1
bound for γ_k	6	5	3	2	2

In the case that $\alpha = (3, 1)$ and $\beta = (2, 2)$ we get

k	1	2	3	4	5	6
max values for γ_k	6	3	2	1	1	1
bound for γ_k	6	4	2	2	1	1

5. Stability: The Kronecker coefficients

In this last section we consider linear upper bounds for $\mathrm{stab}(\alpha, \beta, \gamma)$. Previously known bounds, due to Brion [5] and Vallejo [25] respectively, are

$$M_B(\alpha, \beta; \gamma) = |\alpha| + |\beta| + \gamma_1,$$
$$M_V(\alpha, \beta; \gamma) = |\gamma| + \begin{cases} \max\{|\alpha| + \alpha_1 - 1, |\beta| + \beta_1 - 1, |\gamma|\} & \text{if } \alpha \neq \beta, \\ \max\{|\alpha| + \alpha_1, |\gamma|\} & \text{if } \alpha = \beta. \end{cases}$$

We introduce Lemma 5.1 that produces linear upper bounds for $\mathrm{stab}(\alpha, \beta, \gamma)$ from linear inequalities fulfilled by those (α, β, γ) for which $\bar{g}^{\gamma}_{\alpha,\beta} > 0$. Applying this lemma to different bounds derived in Sections 3 and 4, we obtain two new upper bounds for $\mathrm{stab}(\alpha, \beta, \gamma)$, and recover Brion's bound M_B.

Lemma 5.1. *Let f be a function on triples of partitions such that for all i,*

$$f(\alpha,\beta,\bar{\gamma}) \geqslant f\big(\alpha,\beta,\gamma^{\dagger i}\big).$$

Set $\mathcal{M}_f(\alpha,\beta,\gamma) = |\gamma| + f(\alpha,\beta,\bar{\gamma})$ and assume also that whenever $\bar{g}_{\alpha,\beta}^{\gamma} > 0$,

$$\mathcal{M}_f(\alpha,\beta,\gamma) \geqslant \max\big(|\alpha|+\alpha_1, |\beta|+\beta_1, |\gamma|+\gamma_1\big). \tag{13}$$

Then whenever $\bar{g}_{\alpha,\beta}^{\gamma} > 0$,

$$\mathrm{stab}(\alpha,\beta,\gamma) \leqslant \mathcal{M}_f(\alpha,\beta,\gamma).$$

Proof. Let α, β and γ be partitions such that $\bar{g}_{\alpha,\beta}^{\gamma} > 0$. Let $n \geqslant \mathcal{M}_f(\alpha,\beta,\gamma)$. By Theorem 1.1,

$$g_{\alpha[n]\beta[n]}^{\gamma[n]} = \bar{g}_{\alpha,\beta}^{\gamma} + \sum_{i=1}^{N} (-1)^i \bar{g}_{\alpha,\beta}^{(n-|\gamma|+1,\gamma^{\dagger i})} \tag{14}$$

for some N. Since $n \geqslant \mathcal{M}_f(\alpha,\beta,\gamma) = |\gamma| + f(\alpha,\beta,\bar{\gamma})$, we have $n - |\gamma| + 1 > f(\alpha,\beta,\bar{\gamma})$. Thus $n - |\gamma| + 1 > f(\alpha,\beta,\gamma^{\dagger i})$ for all i. As a consequence, none of the partitions $\tau = (n - |\gamma| + 1, \gamma^{\dagger i})$ fulfills $\mathcal{M}_f(\alpha,\beta,\tau) \geqslant |\tau| + \tau_1$. Indeed, for such a partition, $|\tau| + \tau_1 = |\tau| + (n - |\gamma| + 1)$ and $\mathcal{M}_f(\alpha,\beta,\tau) = |\tau| + f(\alpha,\beta,\gamma^{\dagger i})$. We get that all terms $\bar{g}_{\alpha,\beta}^{(n-|\gamma|+1,\gamma^{\dagger i})}$ in (14) are zero. Therefore $g_{\alpha[n]\beta[n]}^{\gamma[n]}$ is equal to its stable value $\bar{g}_{\alpha,\beta}^{\gamma}$. We conclude that $\mathcal{M}_f \geqslant \mathrm{stab}(\alpha,\beta,\gamma)$. □

Three functions f such that (13) holds have already appeared in this paper. Each one gives a bound for $\mathrm{stab}(\alpha,\beta,\gamma)$.

1. Murnaghan's triangle inequalities (see Murnaghan theorem) and Theorem 4.1 show that (13) holds for $f(\alpha,\beta,\tau) = |\alpha| + |\beta| - |\tau|$. We recover Brion's bound M_B.
2. Theorem 4.1 and Murnaghan's triangle inequalities also imply that (13) holds for $f(\alpha,\beta,\tau) = |\bar{\alpha} \cap \bar{\beta}| + \alpha_1 + \beta_1$. The corresponding bound $\mathcal{M}_f$ is $M_1(\alpha,\beta,\gamma) = |\gamma| + |\bar{\alpha} \cap \bar{\beta}| + \alpha_1 + \beta_1$. Hence, by Lemma 5.1 and the symmetry of the Kronecker coefficients we obtain the proof of Theorem 1.4.
3. Theorem 1.2 shows that (13) holds for $f(\alpha,\beta,\tau) = 1/2(|\alpha| + |\beta| + \alpha_1 + \beta_1 - |\tau|)$, which corresponds to $\mathcal{M}_f = M_2 = \frac{1}{2}(|\alpha| + |\beta| + |\gamma| + \alpha_1 + \beta_1 + \gamma_1)$. The bound $N_2 = [M_2]$ of Theorem 1.5 follows.

Set $N_1(\alpha,\beta,\gamma) = \min\{M_1(\alpha,\beta;\gamma), M_1(\alpha,\gamma;\beta), M_1(\beta,\gamma;\alpha)\}$ and define similarly N_B and N_V from M_B and M_V. Since the Kronecker coefficients $g_{\alpha,\beta}^{\gamma}$ are symmetric under any permutations of the partitions γ,α,β, these are also upper bounds for $\mathrm{stab}(\alpha,\beta,\gamma)$. In the following proposition we show that the bound N_1 improves both Vallejo's N_V and Brion's bound, N_B.

Proposition 5.2. *Let α, β, γ be partitions, then $N_1(\alpha,\beta,\gamma) \leqslant N_B(\alpha,\gamma,\beta)$ and $N_1(\alpha,\beta,\gamma) \leqslant N_V(\alpha,\beta,\gamma)$.*

Proof. For all partitions α, β, γ, we have

$$M_1(\alpha,\beta;\gamma) = |\gamma| + |\bar{\alpha} \cap \bar{\beta}| + \alpha_1 + \beta_1 \leqslant |\gamma| + |\alpha| + \beta_1 = M_B(\alpha,\gamma;\beta),$$

since $|\bar{\alpha} \cap \bar{\beta}| + \alpha_1 \leqslant |\bar{\alpha}| + \alpha_1 = |\alpha|$. This is enough to conclude that $N_1(\alpha,\beta,\gamma) \leqslant N_B(\alpha,\beta,\gamma)$.

We now prove that $N_1(\alpha,\beta,\gamma) \leqslant N_V(\alpha,\beta,\gamma)$. It is enough to prove that for all partitions α, β, γ we have $M_1(\alpha,\beta;\gamma) \leqslant M_V(\alpha,\beta;\gamma)$. By symmetry of both bounds with respect to α and β, we can assume without loss of generality that $\alpha_1 \geqslant \beta_1$. We consider three cases: $\alpha = \beta$; $\alpha \subsetneq \beta$; $\alpha \not\subset \beta$.

If $\alpha = \beta$, then $|\bar\alpha \cap \bar\beta| + \alpha_1 + \beta_1 = |\alpha| + \alpha_1 \leqslant M_V - |\gamma|$.

If $\alpha \subsetneq \beta$, then $|\bar\alpha \cap \bar\beta| + \alpha_1 = |\alpha| \leqslant |\beta| - 1$. Therefore $|\bar\alpha \cap \bar\beta| + \alpha_1 + \beta_1 \leqslant |\beta| + \beta_1 - 1 \leqslant M_V - |\gamma|$.

Finally if $\alpha \not\subset \beta$, then $|\bar\alpha \cap \bar\beta| + \beta_1 = |\alpha \cap \beta| \leqslant |\alpha| - 1$. Therefore $|\bar\alpha \cap \bar\beta| + \alpha_1 + \beta_1 \leqslant |\alpha| + \alpha_1 - 1 \leqslant M_V - |\gamma|$. $\square$

We have shown that N_1 improves the bounds N_B and N_V. In the following two examples we now compare N_2 to N_B and N_V.

Example 3 *(Comparison of N_2 to N_B).* Let $\alpha = (2,1)$ and $\beta = (3,1)$, if $\gamma = (3,1)$, then $N_B = 10$ is greater than $N_2 = 9$ and if $\gamma = (3,2,2)$ then $N_B = 10$ and $N_2 = 11$. This shows that neither one is better than the other.

Example 4 *(Comparison of N_2 to N_V).* Let $\alpha = (2,1)$, $\beta = (3,1)$ and $\gamma = (3,2,2)$, then $N_2 = 11$ and $N_V = 12$, hence $N_2 < N_V$. On the other hand if $\alpha = (3,2)$ and $\beta = (3,1,1)$ and $\gamma = (6)$, then $N_V = 13$ and $N_2 = 14$ and in this case, $N_V < N_2$. This shows that neither N_V nor N_2 is better than the other. Notice that the last example can be generalized as follows. If $|\alpha| = |\beta|$ with $\alpha_1 = \beta_1$ and $\gamma = (\gamma_1)$, then $N_V \leqslant N_2$.

We conclude this section applying our bounds to some interesting examples of Kronecker coefficients appearing in the literature.

Example 5 *(The Kronecker coefficients indexed by three hooks).* Our first example looks at the elegant situation where the three indexing partitions are hooks. Note that after deleting the first part of a hook we always obtain a one column shape. Let $\alpha = (1^e)$, $\beta = (1^f)$ and $\gamma = (1^d)$ be the reduced partitions, with d, e and f positive. In Theorem 3 of [20], it was shown that Murnaghan's inequalities describe the stable value of the Kronecker coefficient $g_{\alpha[n],\beta[n]}^{\gamma[n]}$,

$$\bar g_{\alpha,\beta}^{\gamma} = \big((e \leqslant d+f)\big)\big((d \leqslant e+f)\big)\big((f \leqslant e+d)\big)$$

where $((P))$ equals 1 if the proposition is true, and 0 if not.

Moreover, $\operatorname{stab}(\alpha,\beta,\gamma)$ was actually computed in the proof of Theorem 3 [20]. It was shown that the Kronecker coefficient equals 1 if and only if Murnaghan's inequalities hold, as well as the additional inequality $e + f \leqslant d + 2(n-d) - 2$. This last inequality says that

$$\operatorname{stab}(\alpha,\beta,\gamma) = \left\lceil \frac{d+e+f+3}{2} \right\rceil = N_2(\alpha,\beta,\gamma).$$

To summarize, for triples of hooks, Murnaghan's inequalities govern the value of the reduced Kronecker coefficients, and N_2 is a sharp bound. On the other hand, the bounds provided by N_1, N_B, and N_V are not in general sharp.

Example 6 *(The Kronecker coefficients indexed by two two-row shapes).* After deleting the first part of a two-row partition we obtain a partition of length 1. Let α and β be one-row partitions. We have

$$N_1(\alpha,\beta,\gamma) = \alpha_1 + \beta_1 + \gamma_1,$$

$$N_2(\alpha,\beta,\gamma) = \alpha_1 + \beta_1 + \gamma_1 + \left\lceil \frac{\gamma_2+\gamma_3}{2} \right\rceil.$$

It follows from [3] that when $\bar{g}_{\alpha,\beta}^{\gamma} > 0$,

$$\operatorname{stab}(\alpha, \beta, \gamma) = \gamma_1 - \gamma_3 + \alpha_1 + \beta_1.$$

Neither N_1 nor N_2 are sharp bounds. Indeed, for $\bar{g}_{\alpha,\beta}^{\gamma} > 0$ we have $\operatorname{stab}(\alpha, \beta, \gamma) < N_1$ if $\gamma_3 > 0$, and $\operatorname{stab}(\alpha, \beta, \gamma) < N_2$ if $\gamma_2 > 0$.

Moreover, $N_1 < N_2$ when $\gamma_2 + \gamma_3 > 1$.

Example 7 *(The Kronecker coefficients: One of the partitions is a two-row shape).* The case when γ has only one row, $\gamma = (p)$, was studied in [2]. It was shown there (Theorem 5.1) that

$$\operatorname{stab}\bigl(\alpha, \beta, (p)\bigr) \leqslant |\alpha| + \alpha_1 + 2p.$$

Notice that this bound coincides with $\operatorname{stab}(\alpha, (p))$ after Theorem 1.2. In this case,

$$N_1 = p + |\bar{\alpha} \cap \bar{\beta}| + \alpha_1 + \beta_1,$$

is less than or equal to N_2. It is also mentioned in [2] that, for the case when $\alpha = \beta$, Vallejo's bound N_V does beat this bound (that is, $\operatorname{stab}(\alpha, \alpha)$), but not always. Indeed, when $\alpha = \beta$, N_2 coincides with N_V.

The situation described in the previous example, where $\operatorname{stab}(\alpha, \beta) < N_V(\alpha, \beta, \gamma)$ raises the question of whether $\min(N_1, N_2)$ is always less or equal to $\operatorname{stab}(\alpha, \beta)$ when $\bar{g}_{\alpha,\beta}^{\gamma} > 0$. This is indeed the case since, as a direct consequence of Theorem 3.2, $N_2 \leqslant |\alpha| + |\beta| + \alpha_1 + \beta_1$.

Example 8 *(Vallejo's example).* In [25] the case $\alpha = (3, 2)$, $\beta = (2, 2, 1)$, $\gamma = (2, 2)$ was considered. In this case $\operatorname{stab}(\alpha, \beta, \gamma) = 10$, but

$$N_B(\alpha, \beta, \gamma) = N_V(\alpha, \beta, \gamma) = N_1(\alpha, \beta, \gamma) = 11.$$

Nevertheless, $N_2(\alpha, \beta, \gamma) = 10$.

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